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ELASTIC CONSTANTS AND THERMAL EXPANSION OF CERTAIN BODIES WITH A NONHOMOGENEOUS REGULAR STRUCTURE

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ABSTRACT 24909

A method is presented for determining elastic constants and thermal expansion coefficients of nonhomogeneous bodies. The equations derived may be employed to obtain the formulas for the constants for orthorhombic, tetragonal, and hexagonal structures. They also facilitate a determination of the physico-mechanical characteristics and the stress state of synthetic materials such as plastic glass, in which the binder has elastic, viscous properties.

Author

This article presents a method for determining elastic constants /817* and thermal expansion coefficients of nonhomogeneous bodies, whose regular structure is formed by a biperiodic system of n arbitrarily located hollow cylinders having different diameters. The space between the cylinders is occupied by a medium.

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* Note: Numbers in the margin indicate pagination in the original foreign text.

1. Let E_a, ν_a, α_a and E_s, ν_s, α_s be elastic constants and thermal expansion coefficients of the filler and the medium, $z = x_2 + ix_3$, $\theta = T - T_0$ - temperature, Ω_k' and Ω_k - outer and inner side walls of the cylinders with the centers at the points $a_k + P$ ($P = m\omega_1 + n\omega_2$, $\omega_2 = \omega_1 be^{i\alpha}$, $m, n = 0, \pm 1, \ldots$) (Figure 1). The functions pertaining to the filler are designated by a, those pertaining to the medium are designated by s, and in addition we have:

$$\langle\sigma_{ik}\rangle = F^{-1}\int_F dF\,\sigma_{ik}, \quad \langle\varepsilon_{ik}\rangle = F^{-1}\int_F dF\,\varepsilon_{ik}, \quad F = \omega_1^2 b\sin\alpha. \tag{1}$$

The relationship between $\langle\sigma_{ik}\rangle$ and $\langle\varepsilon_{ik}\rangle$ on areas which are far removed from local perturbations will be as follows, due to the plane of elastic symmetry $x_1 = \text{const}$:

$$\begin{aligned}
\langle\varepsilon_{11}\rangle &= X_{11}\langle\sigma_{11}\rangle + X_{12}\langle\sigma_{22}\rangle + X_{13}\langle\sigma_{33}\rangle + \ldots\ldots + X_{16}\langle\sigma_{23}\rangle + \beta_{11}\theta,\\
\langle\varepsilon_{22}\rangle &= X_{12}\langle\sigma_{11}\rangle + X_{22}\langle\sigma_{22}\rangle + X_{23}\langle\sigma_{33}\rangle + \ldots\ldots + X_{26}\langle\sigma_{23}\rangle + \beta_{22}\theta,\\
\langle\varepsilon_{33}\rangle &= X_{13}\langle\sigma_{11}\rangle + X_{23}\langle\sigma_{22}\rangle + X_{33}\langle\sigma_{33}\rangle + \ldots\ldots + X_{36}\langle\sigma_{23}\rangle + \beta_{33}\theta,\\
\langle\varepsilon_{31}\rangle &= \ldots\ldots\ldots\ldots\ldots\ldots\ X_{44}\langle\sigma_{31}\rangle + X_{45}\langle\sigma_{12}\rangle\ \ldots\ldots,\\
\langle\varepsilon_{12}\rangle &= \ldots\ldots\ldots\ldots\ldots\ldots\ X_{45}\langle\sigma_{31}\rangle + X_{55}\langle\sigma_{12}\rangle\ \ldots\ldots,\\
\langle\varepsilon_{23}\rangle &= X_{16}\langle\sigma_{11}\rangle + X_{26}\langle\sigma_{22}\rangle + X_{36}\langle\sigma_{33}\rangle + \ldots\ldots + X_{66}\langle\sigma_{23}\rangle + \beta_{23}\theta.
\end{aligned} \tag{2}$$

There will be displacements $u_i(z)$ when there is a non-zero shift of the planes (x_1, x_2) and (x_1, x_3). Let Φ_s and $\Phi_{a,k}$ be the holomorphic functions in s and s_k

$$\sigma_{12} - i\sigma_{31} = 2G\Phi(z), \quad u_1 = \mathrm{Re}\int\Phi(z)\,dz. \tag{3}$$

If $\Phi_s(z + \omega_j) = \Phi_s(z)$ $(j = 1,2)$, then the boundary conditions expressing the equality of displacements and stresses on the contact surface must be satisfied on n contours located in one element

$$L_{s,k}\Phi_s - L_{a,k}\Phi_{a,k}|_{\Omega_k'} = 0, \quad l_{a,k}\Phi_{a,k}|_{\Omega_k} = 0, \tag{4}$$ /818

where $L_{s,k}$, $l_{a,k}$ are the differential operators of the boundary conditions;

in addition, we have (Ref. 1):

$$\Phi_s = c_0 - \sum_{k=1}^{n} \sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!} c_{k,s} \zeta^{(s-1)}(z-a_k), \quad \sum_{k=1}^{n} c_{k,1} = 0;$$
$$\Phi_{a,k} = \sum_{m=-\infty}^{\infty} a_{m,k}(z-a_k)^m. \tag{5}$$

On the body surfaces, we define the mean stresses

$$\langle\sigma_{12}\rangle - i\langle\sigma_{31}\rangle = 2G_a\langle\Phi_{a,k}\rangle + 2G_s\langle\Phi_s\rangle. \tag{6}$$

The formulas for elastic constants are obtained by comparing the displacements of the body having a nonhomogeneous structure with the displacements in the body (1) during a displacement (Ref. 2)

$$X_{44}\langle\sigma_{31}\rangle + X_{45}\langle\sigma_{12}\rangle = -2c_0'' + 2\sum_{k=1}^{n}\left\{\delta_2' c_{k,2}' + \left(\delta_1' - \frac{2\pi}{\omega_1^2 b \sin\alpha}\right) c_{k,2}''\right\},$$
$$c_{k,s} = c_{k,s}' + i c_{k,s}'', \tag{7}$$
$$X_{45}\langle\sigma_{31}\rangle + X_{55}\langle\sigma_{12}\rangle = 2c_0' - 2\sum_{k=1}^{n}(\delta_1' c_{k,2}' - \delta_2' c_{k,2}''), \quad \frac{2}{\omega_1}\zeta\left(\frac{\omega_1}{2}\right) = \delta_1' + i\delta_2'.$$

2. When we solve the problem of the body extension by stresses $\langle\sigma_{ii}\rangle = 1$, X_{11}, X_{12}, X_{13}, X_{16} are determined. The complex potentials which satisfy the condition of the stress state periodicity are chosen in the form of the series (5) and

$$\Psi_s = d_0 - \sum_{k=1}^{n}\sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!}\{d_{k,s}\zeta^{(s-1)}(z-a_k) - c_{k,s}\eta^{(s)}(z-a_k)\},$$
$$\Psi_{a,k} = \sum_{m=-\infty}^{\infty} b_{k,m}(z-a_k)^m. \tag{8}$$

Here the elliptic functions of the following form (Ref. 1, 2) are introduced:

$$\eta(z) = \sum_{m,n}{}' \overline{P}\{(z-P)^{-1} + z^2 P^{-3} + z P^{-2}\}.$$

The arbitrary constants are determined from the boundary conditions

$$L_{s,k}(\Phi_s, \Psi_s) - L_{a,k}(\Phi_{a,k}, \Psi_{a,k})|_{\Omega_k'} = f_k(\vartheta), \quad l_{a,k}(\Phi_{a,k}, \Psi_{a,k}) = 0 \tag{9}$$

and from the condition that the principal vector of the forces applied at the elementary element boundary equals zero,

$$\bar{d}_0\bar{\omega}_j-\sum_{k=1}^{n}\bar{d}_{k,2}\bar{\delta}_j=-2c_0\omega_j+\delta_j\sum_{k=1}^{n}c_{k,2}+\bar{\gamma}_j\sum_{k=1}^{n}\bar{c}_{k,2}\ (j=1,\,2). \tag{10}$$

The elastic constants are determined by the formulas

$$\begin{gathered}X_{11}^{-1}=E_a\sum_{k=1}^{n}\xi_k+E_s\eta+8\nu_s(\nu_a-\nu_s)\,G_s\left\{\frac{\nu_a}{\nu_s}\sum_{k=1}^{n}\xi_k a'_{0,k}+\right.\\ \left.+\sum_{k=1}^{n}c_{k,1}\xi'_k|\lambda_k-\sum_{k=1}^{n}\sum_{s=2}^{\infty}\frac{\xi'_k c'_{k,2s}}{2s-1}a_{s-1,0}-\left(\delta'_1-\frac{\pi}{F}\right)\sum_{k=1}^{n}c'_{k,2}+\delta'_2\sum_{k=1}^{n}c''_{k,2}\right\}.\end{gathered} \tag{11}$$

Only the first two terms are dominant terms here /819

$$\nu_{21}=\nu_s+(\nu_s-\nu_a)(\varkappa_s+1)\left(c'_0-\delta'_1\sum_{k=1}^{n}c'_{k,2}+\delta'_2\sum_{k=1}^{n}c''_{k,2}\right),$$

$$\nu_{61}=2(\nu_s-\nu_a)\left\{(\varkappa_s+3)c''_0-(\varkappa_s+1)\left(\delta'_1\sum_{k=1}^{n}c'_{k,2}+\delta'_2\sum_{k=1}^{n}c''_{k,2}\right)\right\},$$

$$\nu_{31}=\nu_{21}+(\nu_s-\nu_a)(\varkappa_s+1)\frac{2\pi}{F}\sum_{k=1}^{n}c'_{k,2},\quad \nu_{ik}=-\frac{X_{ki}}{X_{11}}.$$

3. We can find the remaining constants in (1) by solving the problem of the flat deformed state of the body ($\langle\varepsilon_{ii}\rangle=0$). The form of the solution and the boundary conditions are determined in (4), (8), (9), (10). If we subsequently assume that $\langle\sigma_{22}\rangle=1$, $\langle\sigma_{33}\rangle=\langle\sigma_{23}\rangle=0$, etc., we obtain three systems of algebraic equations, from which c_k and $c_{k,s}$ can be found.

We have the following from the first system, when $\langle\sigma_{22}\rangle=1$, $\langle\sigma_{33}\rangle=$ $=\langle\sigma_{23}\rangle=0$:

$$\begin{gathered}X_{22}=-\nu_{21}X_{12}+\frac{\varkappa_s+1}{2G_s}\left\{\frac{1}{4}+c'_0-\delta'_1\sum_{k=1}^{n}c'_{k,2}+\delta'_2\sum_{k=1}^{n}c''_{k,2}\right\},\\ X_{23}=X_{22}-\frac{1}{2G_s}+\nu_{21}(X_{12}-X_{13})+(\varkappa_s+1)\frac{\pi}{G_sF}\sum_{k=1}^{n}\ddot{c}_{k,2}.\end{gathered} \tag{12}$$

The following is derived from the second system in the case of $\langle\sigma_{33}\rangle = 1$, $\langle\sigma_{22}\rangle = \langle\sigma_{23}\rangle = 0$

$$X_{33} = -\nu_{31}X_{13} + \frac{\varkappa_s + 1}{2G_s}\left\{\frac{1}{4} + c_0' + \delta_2' \sum_{k=1}^{n} c_{k,3}'' - \left(\delta_1' - \frac{2\pi}{F}\right)\sum_{k=1}^{n} c_{k,3}'\right\}, \tag{13}$$

$$X_{36} = -\nu_{31}X_{16} + \frac{1}{G_s}\left\{(\varkappa_s + 3)c_0'' - (\varkappa_s + 1)\left(\delta_1' \sum_{k=1}^{n} c_{k,3}'' + \delta_2' \sum_{k=1}^{n} c_{k,3}'\right)\right\}.$$

In the case of $\langle\sigma_{23}\rangle = 1$, we have the following from the third system

$$X_{36} = -\nu_{61}X_{13} + \frac{\varkappa_s + 1}{2G_s}\left\{c_0 - \delta_1' \sum_{k=1}^{n} c_{k,3}' + \delta_2' \sum_{k=1}^{n} c_{k,3}''\right\},$$

$$X_{66} = -\nu_{61}X_{16} + \frac{1}{G_s}\left\{1 + (\varkappa_s + 3)c_0'' - (\varkappa_s + 1)\left(\delta_1' \sum_{k=1}^{n} c_{k,3}'' + \delta_2' \sum_{k=1}^{n} c_{k,3}'\right)\right\}. \tag{14}$$

Here $\xi_k = \pi(\lambda_k^2 - \varepsilon_k^2) / F$, $\xi_k' = \pi\lambda_k^2 / F$, $\varkappa = 3-4\nu$, $\alpha_{i,k}$ are expansion coefficients in $\xi^{(k)}(z)$ Laurent series.

4. For increased temperature θ, due to stress redistribution the transverse cross sections x_1 = const remain flat as one recedes from the body edge. Therefore, we have

$$\langle\varepsilon_{11}\rangle = \alpha_s\theta + \langle\varepsilon_{11}\rangle_s = \alpha_a\theta + \langle\varepsilon_{11}\rangle_a. \tag{15}$$

The over-all solution in this case is comprised of the solution for the problem of body extension by stresses $\langle\sigma_{11}\rangle$ without allowance for the interaction between the filler and the medium, as well as the solution taking this interaction into account. The complex potentials have the form (5) and (8). The tensor components of thermal expansion are found in the following form: /820

$$\beta_{11} = \alpha_s - (\alpha_s - \alpha_a)\frac{X_{11}}{\nu_s - \nu_a}\left\{(1 + \nu_s)E_a \sum_{k=1}^{n} \xi_k - (1 + \nu_a)(X_{11}^{-1} - \eta E_s)\right\}, \tag{16}$$

$$\beta_{22} = \alpha_s + (\alpha_s - \beta_{11})\nu_{21} - (\alpha_s - \alpha_a)(1 + \nu_a)\frac{\nu_s - \nu_{21}}{\nu_s - \nu_a},$$

$$\beta_{33} = \alpha_s + (\alpha_s - \beta_{11})\nu_{31} - (\alpha_s - \alpha_a)(1 + \nu_a)\frac{\nu_s - \nu_{31}}{\nu_s - \nu_a}, \qquad (16)\ \text{(cont.)}$$

$$\beta_{23} = \nu_{61}\left\{\alpha_s - \beta_{11} + (\alpha_s - \alpha_a)\frac{1 + \nu_a}{\nu_s - \nu_a}\right\}.$$

The equations given above can be readily employed to obtain the formulas for the constants in the case of the simpler structures - orthorhombic, tetragonal and hexagonal.

The formulas obtained are suitable for determining the physico-mechanical characteristics and the stress state of synthetic materials like plastic glass, in which the binder has elastic, viscous properties, and only the constants E_s, ν_s must be replaced by the corresponding linear operators.

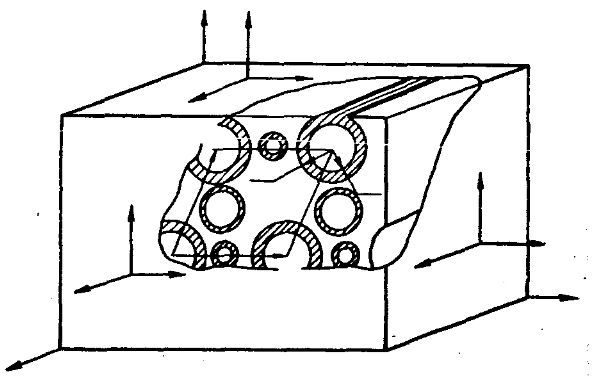

Figure 1

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